Coins: Assumptions and Results

Part-2

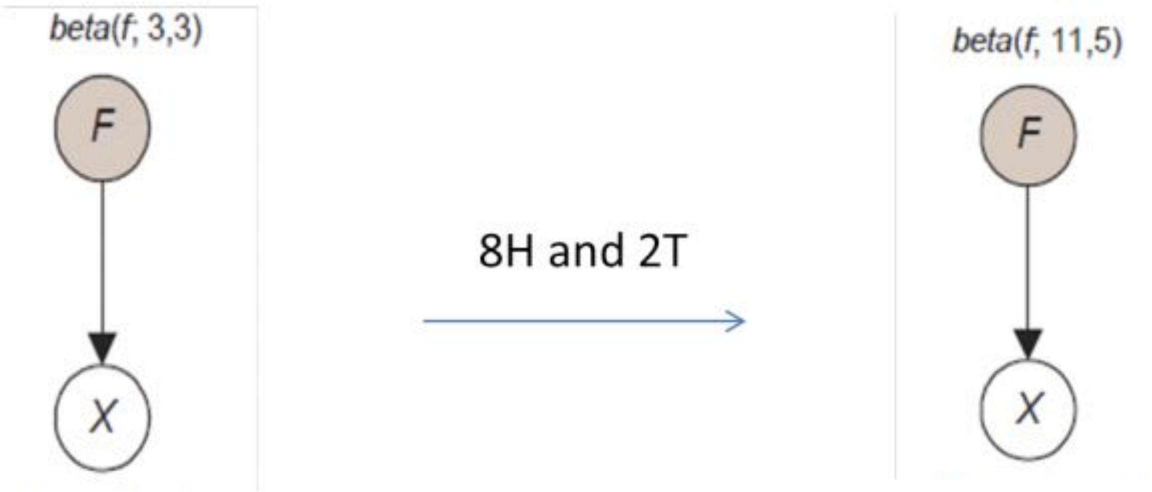
20th March, 2020

Single Coin Problem

Prior: Given a single coin with a prior p.d.f of heads specified by $F \sim beta(f\ ; \alpha, \beta)$, probability of the outcome on the first toss

$$P(X = 1) = E(F) = \alpha/(\alpha + \beta)$$

Posterior: Given a single coin with a prior p.d.f for heads specified by $F \sim beta(f\ ; \alpha, \beta)$, the prior probability of heads after obtaining data consisting of **s** heads and **t** tails is $F|D \sim beta(f\ ; \alpha + s, \beta + t)$



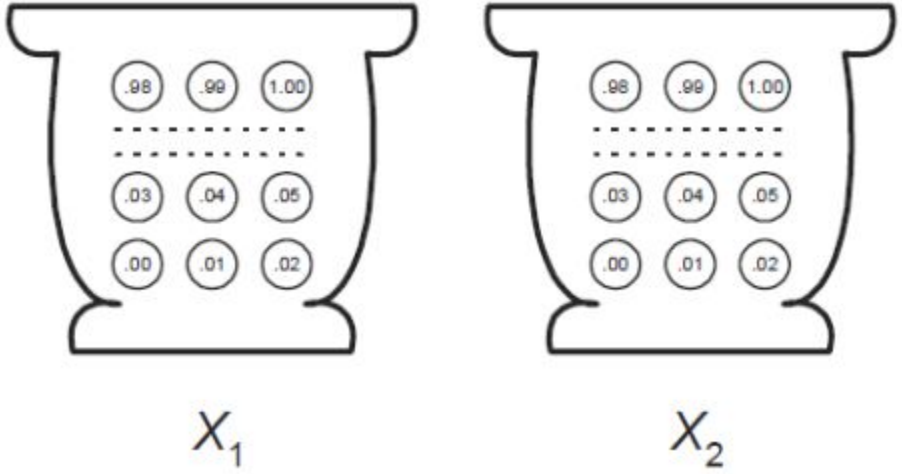
Also, the posterior probability is given by:

$$P(X = 1|d) = E(F|d) = (\alpha + s) / (\alpha + \beta + s + t)$$

Likelihood: Given a single coin with a prior p.d.f of heads specified by *F ~ beta(f ; α, β)*, the probability of data **d** consisting of **s** heads and **t** tails is

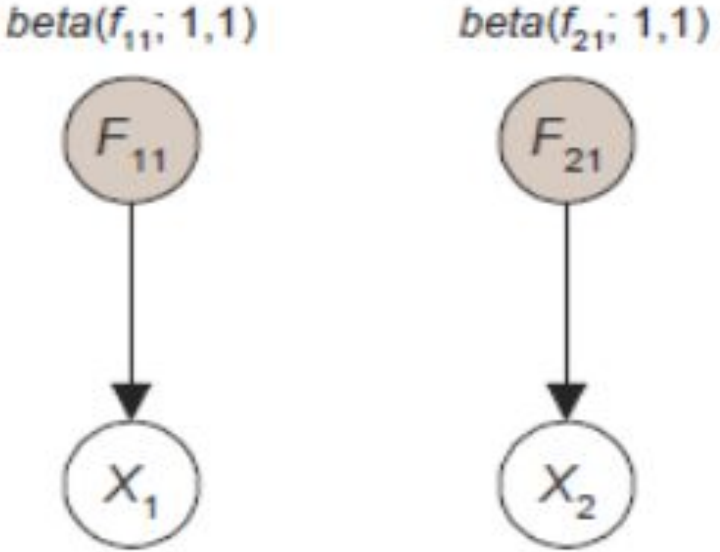
$$P(d) = \frac{B(\alpha + s, \beta + t)}{B(\alpha, \beta)}$$

Two URN Problem



- **$P(X_1 = 1, X_2 = 1) = P(X_1 = 1) \times P(X_2 = 1)$**

- Assume the (joint) prior probability distribution for sampling and tossing coins from the urn can be represented by the following graph:



Q1(a). Using the graphical model of joint probability shown, what is $P(X_1 = 1, X_2 = 1)$?

Ans. For the graphical model shown, $P(X_i = 1) = E(F_{i1})$.

$$E(F_{i1}) = 1/(1 + 1) = ½$$

Using the fact that **X_1 and X_2 are independent** of each other we get,

$$P(X_1 = 1, X_2 = 1) = P(X_1 = 1)P(X_2 = 1) = 1/4.$$

Q1(b). We observe the data d below from 7 tosses of two binary random variables X_1 and X_2:

Case	X_1	X_2
1	1	1
2	1	1
3	1	1
4	1	2
5	2	1
6	2	1
7	2	2

Assume the variables are are independent of each other, and that prior p.d.f.'s over the probabilities of 1's is as shown previously. What is the probability $P(X_1 = 1, X_2 = 1 \mid d)$?

Ans. Modelling the data as being generated by drawing coins from independent urns, then after data from the 7 tosses, the posteriors on the (relative) frequencies of heads are **beta(f_{11}; 1 + 4, 1 + 3)** and **beta(f_{21}; 1 + 5, 1 + 2)**.

$$P(X_1 = 1, X_2 = 1 \mid d) = P(X_1 = 1|d)\, P(X_2 = 1|d)$$

$$= E(F_{11}|d)\, E(F_{21}|d)$$

$$= (5/9) * (6/9)$$

$$= (10/27) = 0.37$$

Q1(c). What is P(d)?

Ans. Since the coins are independent, we can treat the probabilities of obtaining 1's and 2's from them as independent events. Thus,

P(d) = P(4H, 3T from X1 and 5H, 2T from X2)

= P(4H, 3T from X1)P(5H, 2T from X2)

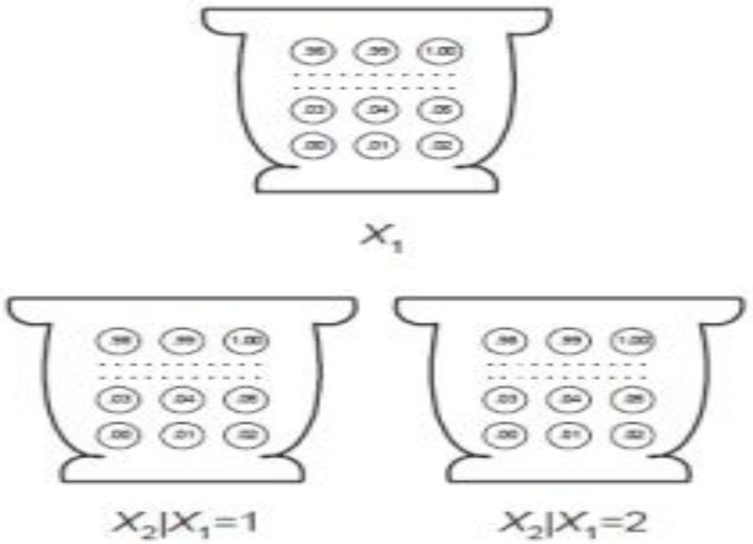
As we know,

$$P(d) = \frac{B(\alpha + s, \beta + s)}{B(\alpha, \beta)}$$

Hence, P(d) = (B(1 + 4, 1 + 3) / B(1, 1)) * (B(1 + 5, 1 + 2) / B(1, 1))

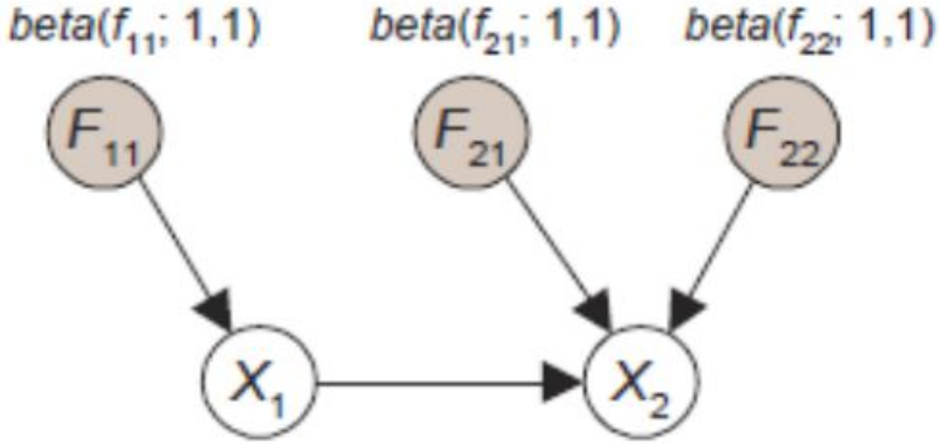
$\approx 2 \times 10^{-5}$

Conditional URN Problem



SAMPLING RULE: We draw a coin from Urn X_1 and toss it. If it lands **heads**, we draw a coin from **Urn $X_2|X_1$ = 1** otherwise we draw a coin from **Urn $X_2|X_1$ = 2**

There are 3 kinds of coin-tosses **(corresponding to coins drawn from the 3 urns)**: $\mathbf{X_1}$, $\mathbf{X_2|X_1 = 1}$ and $\mathbf{X_2|X_1 = 2}$. Each coin has its own prior p.d.f's. The graph structure describing this is:



F_{11} -> p.d.f of first coin landing heads

F_{21} -> p.d.f. of the second coin landing heads given the first coin lands heads.

F_{22} -> p.d.f. of the second coin landing heads given the first coin lands tails.

Q2(a). For the p.d.f's shown in the graph earlier, calculate the 4 joint probabilities of X1, X2.

Ans. $P(X_1 = 1, X_2 = 1) = P(X_1 = 1)P(X_2 = 1|X_1 = 1)$

$$= E(F_{11})E(F_{21}) = ½ * ½ = ¼$$

Similarly,

$P(X_1 = 1, X_2 = 2) = ¼$

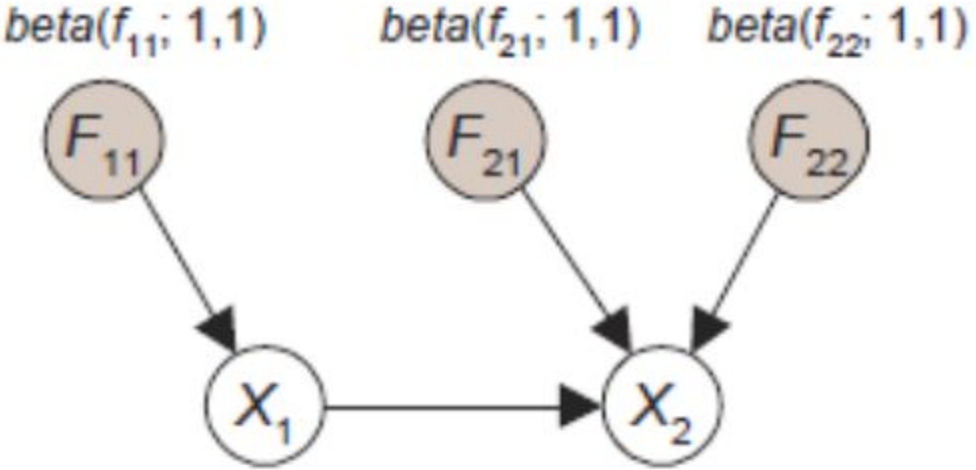
$P(X_1 = 2, X_2 = 1) = ¼$

$P(X_1 = 2, X_2 = 2) = ¼$

Q2(b). We observe the data d below from 7 tosses of two binary random variables X1 and X2:

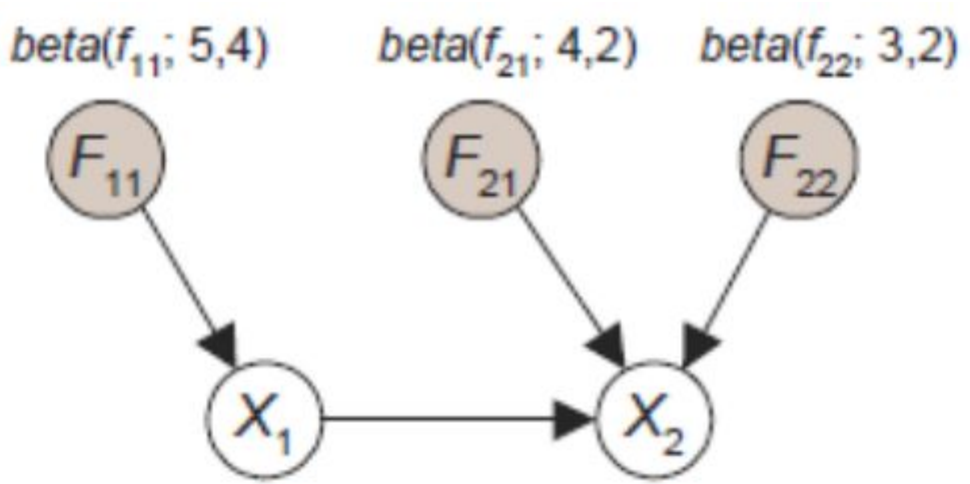
Case	X_1	X_2
1	1	1
2	1	1
3	1	1
4	1	2
5	2	1
6	2	1
7	2	2

Assume the variable X_2 is conditionally dependent on X_1, and that prior p.d.f.'s over the probabilities of 1's is as shown in the next slide:



Graphically show the posterior distributions of the Fij in the network, and find **$P(X_1 = 1)$**, **$P(X_2 = 1|X_1 = 1)$** and **$P(X_2 = 1|X_1 = 2)$** using the posterior distributions.

Ans.



P(X1 = 1) = E(F11|d) = 5/9

P(X2 = 1|X1 = 1) = E(F21|d) = 4/6

P(X2 = 1|X1 = 2) = E(F22|d) = 3/5

Q2(c). Calculate P(d).

Ans. P(d) = P(4H, 3T from X_1 and 5H, 2T from X_2)

= P(4H, 3T from X_1) * P(3H, 1T from X_2 | X_1 = 1) * P(2H, 1T from X_2 | X_1 = 2)

$$P(d) = \frac{B(5,4)}{B(1,1)} \frac{B(4,2)}{B(1,1)} \frac{B(3,2)}{B(1,1)}$$

- In general, for each variable X, we will get a single "coin" for each instantiation (combination) of values of the parent variables in the graph (by convention, we will take variables that have no parents as having a single instantiation)
- Let there are **n** variables, and a variable $\mathbf{X_i}$ have $\mathbf{n_i}$ instantiations. Suppose the data result in $\mathbf{s_{ij}}$ heads for the $\mathbf{j^{th}}$ instantiation of the parents of $\mathbf{X_i}$; and $\mathbf{t_{ij}}$ the corresponding number of of tails. Then

$$P(d) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} \frac{B(\alpha_{ij} + s_{ij}, \beta + t_{ij})}{B(\alpha_{ij}, \beta_{ij}}$$

THANK YOU!