Structural Learning in BN

27th March, 2020

Revision

For coins with Beta priors:

- **Prior:** P(X = 1) = E(F) = α/(α + β)
- **Posterior:** P(X = 1|d) = E(F|d) = (α + s) / (α + β + s + t)
- **Likelihood:** $P(d) = \frac{B(\alpha + s, \beta + s)}{B(\alpha, \beta)}$ **; where** $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$
- **Generalised Formula:**

$$P(d) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} \frac{B(\alpha_{ij} + s_{ij}, \beta + t_{ij})}{B(\alpha_{ij}, \beta_{ij}}$$

- **Previous Assumption:** The graph structure (DAG - Directed Acyclic Graph) was known to us.
- Hence P(d) was P(d|G) actually.
- For multiple graphs $\{G_1, G_2, . . . , G_n\}$, P(d) can be calculated as:

$$P(d) = P(d|G_1)*P(G_1) + P(d|G_2)*P(G_2) + . . . + P(d|G_n)*P(G_n)$$

If we have several potential DAG structures G_1 , G_2 , . . . , G_n we can use Bayes' theorem to calculate:

$$P(G_i |D) = \alpha P(D|G_i)P(G_i)$$

And then choose the Graph which has maximum probability given the data D.

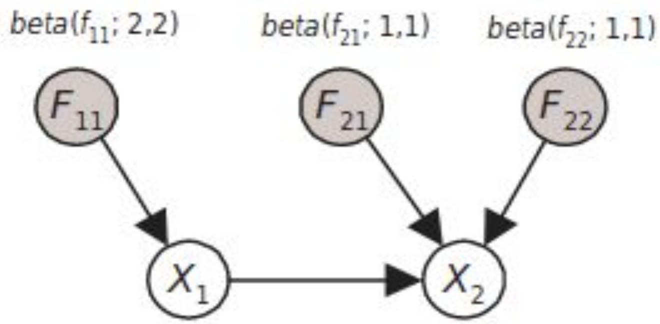
Ex1: Suppose we are doing a study concerning individuals who were married by age 30, and we want to see if there is a correlation between graduating college and getting divorced.

Variable	Value	When the Variable Takes this Value
X_1	1	*Individual graduated college*
	2	*Individual did not graduate college*
X_2	1	*Individual was divorced by age 50*
	2	*Individual was not divorced by age 50*

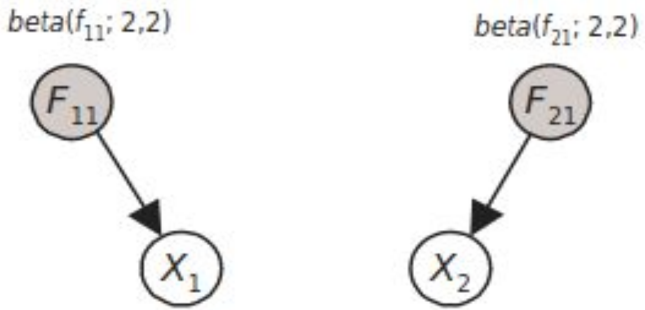
Next, we observe following data:

Case	X_1	X_2
1	1	1
2	1	2
3	1	1
4	2	2
5	1	1
6	2	1
7	1	1
8	2	2

Now, given two potential graphs, find out which graph is more probable.



gp_1



gp_2

Ans.

Step1: Calculate $P(d|G_i)$; here $i \in \{1,2\}$.

$$P(\mathrm{d}|gp_1) = \left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+5)\Gamma(2+3)}{\Gamma(2)\Gamma(2)}\right)\left(\frac{\Gamma(2)}{\Gamma(2+5)}\frac{\Gamma(1+4)\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\right)\left(\frac{\Gamma(2)}{\Gamma(2+3)}\frac{\Gamma(1+1)\Gamma(1+2)}{\Gamma(1)\Gamma(1)}\right)$$

$$= 7.2150\times10^{-6}$$

$$P(\mathrm{d}|gp_2) = \left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+5)\Gamma(2+3)}{\Gamma(2)\Gamma(2)}\right)\left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+5)\Gamma(2+3)}{\Gamma(2)\Gamma(2)}\right)$$

$$= 6.7465\times10^{-6}$$

Step2:

Assume that our prior belief is that neither model (gp_1, gp_2) is more probable than the other, hence we say that each model is equally likely.

$$P(gp_1) = P(gp_2) = 0.5$$

Step3: Using Bayes' theorem:

$$P(gp_1|\mathrm{d}) = \frac{P(\mathrm{d}|gp_1)P(gp_1)}{P(\mathrm{d})}$$

$$= (\ 7.2150\times10^{-6}*(0.5)\ )\ /\ P(d)$$

$$= \alpha(3.607\ 5\times10-6) \qquad \text{Here } \alpha = 1/P(d)$$

Similarly,

$$P(gp_2|\mathrm{d}) = \frac{P(\mathrm{d}|gp_1)P(gp_1)}{P(\mathrm{d})}$$

$= (6.746\ 5\times10^{-6}(.5)) / P(d)$

$= \alpha(3.373\ 25\times10^{-6})$

Eliminating α we get,

$P(gp_1|d) = 0.51678$

$P(gp_2|d) = 0.48322$

Hence, we conclude it is more probable that college attendance and divorce are correlated.

Q1. Suppose for the same potential graph models gp_1 and gp_2 as given in the previous example the observed data is the following:

Case	X_1	X_2
1	1	1
2	1	1
3	1	1
4	1	1
5	2	2
6	2	2
7	2	2
8	2	2

Find the probabilities of these DAG's occurring given the above mentioned data. Also, prior beliefs for both of these graphs are equal.

Ans.

$$P(\mathrm{d}|gp_1) = \left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+4)\Gamma(2+4)}{\Gamma(2)\Gamma(2)}\right)\left(\frac{\Gamma(2)}{\Gamma(2+4)}\frac{\Gamma(1+4)\Gamma(1+0)}{\Gamma(1)\Gamma(1)}\right)\left(\frac{\Gamma(2)}{\Gamma(2+4)}\frac{\Gamma(1+0)\Gamma(1+4)}{\Gamma(1)\Gamma(1)}\right)$$

$= \alpha(8.6580\times10^{-5})$

$$P(\mathrm{d}|gp_2) = \left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+4)\Gamma(2+4)}{\Gamma(2)\Gamma(2)}\right)\left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+4)\Gamma(2+4)}{\Gamma(2)\Gamma(2)}\right)$$

$= \alpha(4.6851\times10^{-6})$

Using Bayes' theorem,

$P(gp_1|d) = 0.94866$

$P(gp_2|d) = 0.05134$

Hence gp_1 is more probable.

Q2. Suppose for the same potential graph models gp_1 and gp_2 as given in the previous example the observed data is the following:

Case	X_1	X_2
1	1	1
2	1	1
3	1	2
4	1	2
5	2	1
6	2	1
7	2	2
8	2	2

Find the probabilities of these DAG's occurring given the above mentioned data. Also, prior beliefs for both of these graphs are equal.

Ans.

$$P(\mathrm{d}|gp_1) = \left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+4)\Gamma(2+4)}{\Gamma(2)\Gamma(2)}\right)\left(\frac{\Gamma(2)}{\Gamma(2+4)}\frac{\Gamma(1+2)\Gamma(1+2)}{\Gamma(1)\Gamma(1)}\right)\left(\frac{\Gamma(2)}{\Gamma(2+4)}\frac{\Gamma(1+2)\Gamma(1+2)}{\Gamma(1)\Gamma(1)}\right)$$

$$= \alpha(2.4050\times10^{-6})$$

$$P(\mathrm{d}|gp_2) = \left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+4)\Gamma(2+4)}{\Gamma(2)\Gamma(2)}\right)\left(\frac{\Gamma(4)}{\Gamma(4+8)}\frac{\Gamma(2+4)\Gamma(2+4)}{\Gamma(2)\Gamma(2)}\right)$$

$$= \alpha(4.6851\times10^{-6})$$

Using Bayes' theorem,

$P(gp_1|d) = 0.33921$

$P(gp_2|d) = 0.66079$

Hence gp_2 is more probable.

THANK YOU!